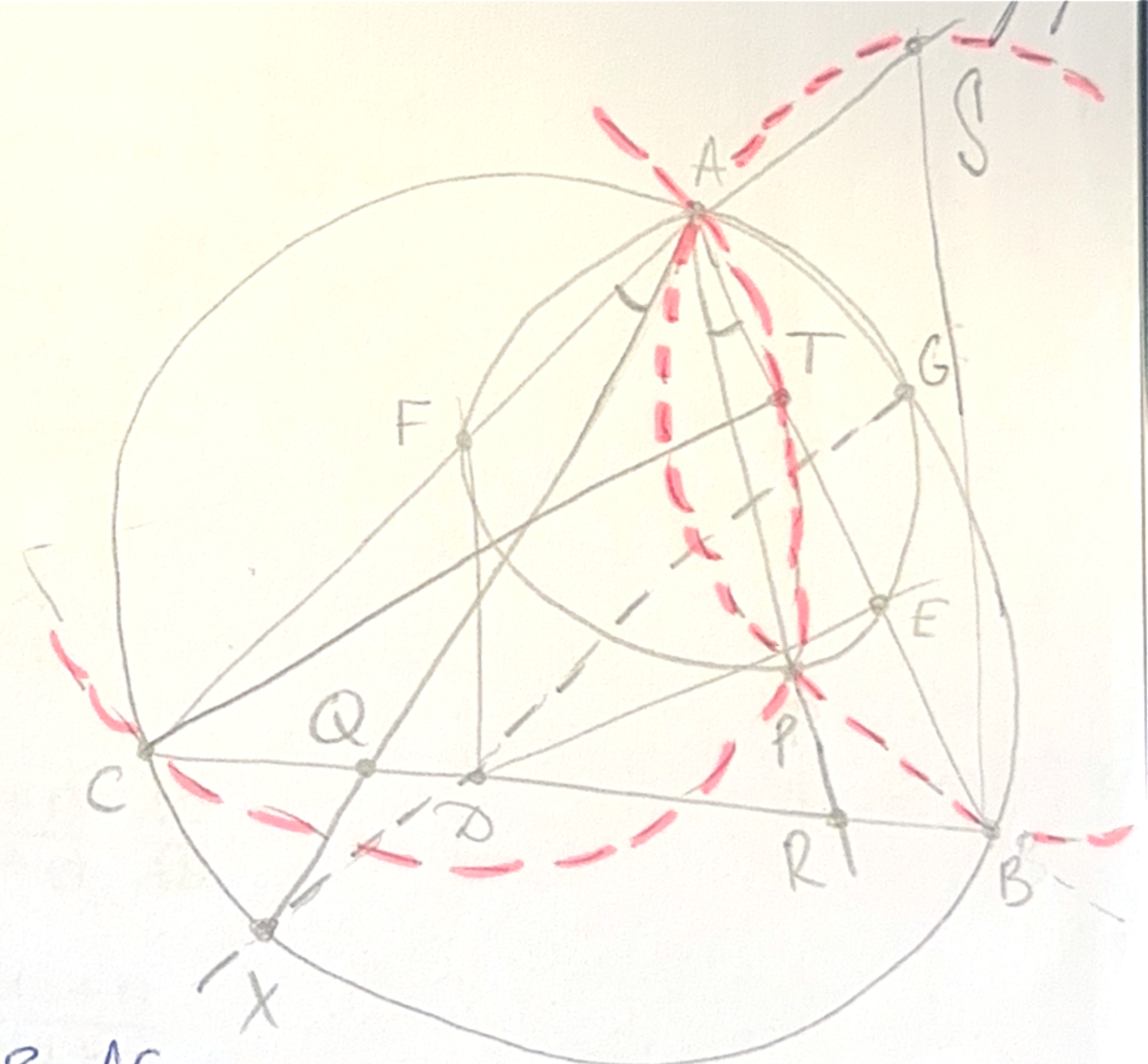
(Generalized APL)

Let Q be on BC such that,

$$\frac{BQ}{CQ} = \frac{AB}{AC} \cdot \frac{CS}{BT}$$

then, let $X = AQ \cap BC$ then we want to show,

$$\boxed{(B, C; D, Q) \overset{X}{\underset{?}{=}} (B, C; G, A)}$$



notice since G is Miquel $(BCFE)$, then,

$$(B, C; G, A) = \frac{GB/GC}{AB/AC} = \frac{EB/CF}{AB/AC} = \frac{EB \cdot AC}{AB \cdot FC}$$

but $(B, C; D, Q) = \dfrac{BD/CD}{BQ/CQ}$ thus it is all equivalent to, $\dfrac{BQ}{CQ} = \dfrac{BD}{CD} \cdot \dfrac{AB \cdot FC}{EB \cdot AC}$,

however

$$\frac{BD}{\cancel{CD}\ EB} \cdot \frac{AB}{AC} \cdot \frac{FC}{CD} = \frac{AB}{AC} \cdot \frac{BC}{BT} \cdot \frac{CS}{BC} = \frac{AB}{AC} \cdot \frac{CS}{BT} \quad \blacksquare$$

By Coaxility lemma the circle (AEF) passes through a const. point.

(defining $P = (ABS) \cap (ACT)$ by $\boxed{\text{quick}}$ bary-bash we get that AP is isogonal to AQ)

§3 Barycentric Bash for APL

Theorem 3.1 *AP is isogonal to AQ in $\angle CAB$*

Proof. Let $T = (s : t : 0)$ and $S = (p : 0 : q)$, then let us find the formula for the circumcircle of (ABS),

$$-a^2yz - b^2xz - c^2xy + (x+y+z)(ux+vy+wz) = 0$$

$$\begin{cases} A = (1:0:0), & u = 0 \\ B = (0:1:0), & v = 0 \\ S = (x:0:z), & -b^2pq + (p+q)\cdot wq = 0 \end{cases}$$

thus,

$$b^2p = (p+q)w \implies w = \frac{b^2p}{p+q}$$

Thus,

$$-a^2yz - b^2xz - c^2xy + (x+y+z)\cdot\frac{b^2p\cdot z}{p+q} = 0, \text{ for } (ABS)$$

similarly for (ACT), we obtain,

$$-a^2yz - b^2xz - c^2xy + (x+y+z)\cdot\frac{c^2s\cdot y}{s+t} = 0$$

now let us intersect these two circles,

$$\begin{cases} -a^2yz - b^2xz - c^2xy + (x+y+z)\cdot\frac{b^2p\cdot z}{p+q} = 0 \\ -a^2yz - b^2xz - c^2xy + (x+y+z)\cdot\frac{c^2s\cdot y}{s+t} = 0 \end{cases}$$

$$\implies (x+y+z)\left(\frac{b^2p\cdot z}{p+q} - \frac{c^2s\cdot y}{s+t}\right) = 0$$

$$\frac{b^2p\cdot z}{p+q} = \frac{c^2s\cdot y}{s+t}$$

thus we can make the following assumptions,

$$\begin{cases} y = \frac{b^2p}{p+q} \\ z = \frac{c^2s}{s+t} \end{cases}$$

with unhomogenized coordinates. Notice,

$$\frac{CS}{BT} = \frac{p(s+t)}{s(p+q)}\cdot\frac{AC}{AB}$$

thus,

$$\frac{BQ}{CQ} = \frac{p(s+t)}{s(p+q)}$$

but the isogonal of AR is given by,

$$\frac{BR^*}{CR^*} = \frac{RC}{RB}\cdot\frac{c^2}{b^2} = \left(\frac{b^2p}{p+q}\right)/\left(\frac{c^2s}{s+t}\right)\cdot\frac{c^2}{b^2} = \frac{p(s+t)}{s(p+q)}$$

thus we are done! □